Lec5 Transmission Lines and waveguides

3.1 GENERAL SOLUTIONS FOR TEM, TE, AND TM WAVES

We assume time-harmonic fields with an $e^{j\omega t}$ dependence and wave propagation along the z-axis. The electric and magnetic fields can then be written as

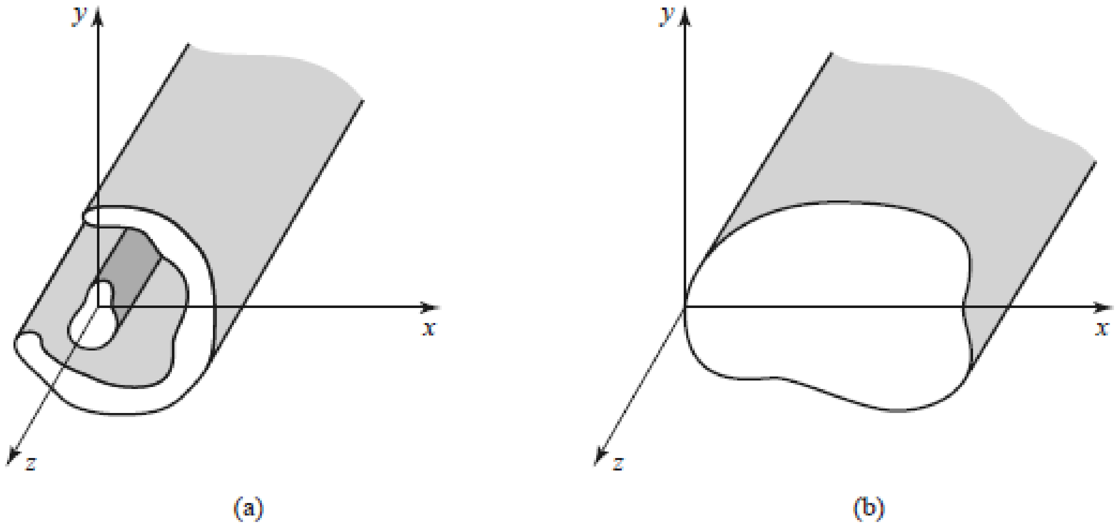


FIGURE 3.1 (a) General two-conductor transmission line and (b) closed waveguide.

$$\bar{E}(x, y, z) = [\bar{e}(x, y) + \hat{z}e_z(x, y)]e^{-j\beta z},$$

$$\bar{H}(x, y, z) = [\bar{h}(x, y) + \hat{z}h_z(x, y)]e^{-j\beta z},$$

e(x, y) and h(x, y) represent the transverse electric and magnetic field components, and ez and hz are the longitudinal electric and magnetic field components.

if conductor or dielectric loss is present, the propagation constant will be complex; $j\beta$ should then be replaced with $\gamma = \alpha + j\beta$.

Assuming that the transmission line or waveguide region is source free,

$$\nabla \times \bar{E} = -j\omega\mu\bar{H},$$
$$\nabla \times \bar{H} = j\omega\epsilon\bar{E}.$$

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x,$$
$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y,$$
$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z,$$
$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x,$$
$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y,$$
$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z.$$

$$H_x = \frac{j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - \beta\frac{\partial H_z}{\partial x}\right),$$
$$H_y = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + \beta\frac{\partial H_z}{\partial y}\right),$$
$$E_x = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right),$$
$$E_y = \frac{j}{k_c^2}\left(-\beta\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right),$$

the cutoff wave number;

$$k_c^2 = k^2 - \beta^2$$

$$k = \omega\sqrt{\mu\epsilon} = 2\pi/\lambda$$

TEM Waves

Transverse electromagnetic (TEM) waves are characterized by Ez = Hz = 0.

if Ez = Hz = 0, then the transverse fields are also all zero, unless $k_c^2 = 0$ ($k^2 = \beta^2$).

$$\nabla^2 \bar{E} + \omega^2 \mu \epsilon \bar{E} = 0,$$

The Helmholtz wave equation for *Ex is*,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_x = 0,$$

$$(\partial^2/\partial z^2) E_x = -\beta^2 E_x = -k^2 E_x$$

$$\rightarrow \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) E_x = 0.$$

A similar result also applies to *Ey*, then $\nabla_t^2 \bar{e}(x, y) = 0,$

$\nabla_t^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian operator in the two transverse dimensions.

It is easy to show in the same way that the transverse magnetic fields also satisfy Laplace's equation:

$$\nabla_t^2 \bar{h}(x, y) = 0.$$

The transverse fields of a TEM wave are thus the same as the static fields that can exist between the conductors.

In the electrostatic case, we know that the electric field can be expressed as the gradient of a scalar potential

$$\bar{e}(x, y) = -\nabla_t \Phi(x, y),$$

$\nabla_t = \hat{x}(\partial/\partial x) + \hat{y}(\partial/\partial y)$ is the transverse gradient operator in two dimensions.

the curl of $\bar{e}$ must vanish,

$$\nabla_t \times \bar{e} = -j\omega\mu h_z \hat{z} = 0.$$

Using the fact that $\nabla \cdot \bar{D} = \epsilon \nabla_t \cdot \bar{e} = 0$ $\quad \nabla_t^2 \Phi(x, y) = 0,$

The voltage between two conductors can be found as

$$V_{12} = \Phi_1 - \Phi_2 = \int_1^2 \bar{E} \cdot d\bar{\ell},$$

where Φ_1 and Φ_2 represent the potential at conductors 1 and 2, respectively.

The current flow on a given conductor can be found from Ampere's law as

$$I = \oint_C \bar{H} \cdot d\bar{\ell},$$

where C is the cross-sectional contour of the conductor.

- **TEM waves can exist when two or more conductors are present.**
- **Plane waves are also examples of TEM waves, in this case the transmission line conductors may be considered to be two infinitely large plates separated to infinity.**
- **A closed conductor (such as a rectangular waveguide) cannot support TEM waves since the corresponding static potential in such a region would be zero (or possibly a constant),** leading to $\bar{e} = 0$.

The wave impedance of a TEM mode can be found as the ratio of the transverse electric and magnetic fields:

$$Z_{\text{TEM}} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \sqrt{\frac{\mu}{\epsilon}} = \eta,$$

The other pair of transverse field components gives

$$Z_{\text{TEM}} = \frac{-E_y}{H_x} = \sqrt{\frac{\mu}{\epsilon}} = \eta.$$

$$\bar{h}(x, y) = \frac{1}{Z_{\text{TEM}}}\hat{z} \times \bar{e}(x, y).$$

Compare wave impedance and characteristic impedance

Characteristic impedance relates traveling voltage and current and is a function of the line geometry as well as the material filling the line.

Wave impedance relates transverse field components and is dependent only on the material constants.

The procedure for analyzing a TEM line can be summarized as follows:

1. Solve Laplace's equation $\nabla_t^2 \Phi(x, y) = 0,$

2. Find these constants by applying the boundary conditions for the known voltages on the conductors.

3. Compute $\bar{e}(x, y) = -\nabla_t \Phi(x, y),$ $\bar{E}(x, y, z) = [\bar{e}(x, y) + \hat{z} e_z(x, y)] e^{-j\beta z},$

$$\bar{h}(x, y) = \frac{1}{Z_{\text{TEM}}} \hat{z} \times \bar{e}(x, y). \qquad \bar{H}(x, y, z) = [\bar{h}(x, y) + \hat{z} h_z(x, y)] e^{-j\beta z},$$

4. Compute V and I $V_{12} = \Phi_1 - \Phi_2 = \int_1^2 \bar{E} \cdot d\bar{\ell},$ $I = \oint_C \bar{H} \cdot d\bar{\ell},$

5. The propagation constant $\beta = \omega\sqrt{\mu\epsilon} = k,$

the characteristic impedance is given by $Z_0 = V/I$.

TE Waves

Transverse electric (TE) waves, (also referred to as H-waves) are characterized by **Ez = 0** and. $H_z \neq 0$.

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x,$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y,$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z,$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x,$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y,$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z.$$

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x},$$

$$H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y},$$

$$E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y},$$

$$E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}.$$

In this case $k_c \neq 0$, and the propagation constant $\beta = \sqrt{k^2 - k_c^2}$

is generally a function of frequency and the geometry of the line or guide.

from the Helmholtz wave equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) H_z = 0,$$

since $H_z(x, y, z) = h_z(x, y)e^{-j\beta z}$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z = 0,$$

The TE wave impedance is

$$Z_{\mathrm{TE}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta},$$

which is seen to be frequency dependent. TE waves can be supported inside closed conductors, as well as between two or more conductors.

TM Waves

Transverse magnetic (TM) waves (also referred to as E-waves) are characterized by Ez ≠ 0 and Hz = 0.

$$\frac{\partial E_z}{\partial y} + j\beta E_y = -j\omega\mu H_x,$$

$$-j\beta E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y,$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z,$$

$$\frac{\partial H_z}{\partial y} + j\beta H_y = j\omega\epsilon E_x,$$

$$-j\beta H_x - \frac{\partial H_z}{\partial x} = j\omega\epsilon E_y,$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\epsilon E_z.$$

→

$$H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y},$$

$$H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x},$$

$$E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x},$$

$$E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}.$$

In this case $k_c \neq 0$, and the propagation constant $\beta = \sqrt{k^2 - k_c^2}$

is generally a function of frequency and the geometry of the line or guide.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right) E_z = 0,$$

since $E_z(x, y, z) = e_z(x, y)e^{-j\beta z}$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z = 0,$$

The TM wave impedance is

$$Z_{\text{TM}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k},$$

which is frequency dependent.

TM waves can be supported inside closed conductors, as well as between two or more conductors.

The procedure for analyzing TE and TM waveguides :

1. Solve the reduced Helmholtz equation, for *hz or ez . The solution* will contain several unknown constants and the unknown cutoff wave number, *kc*.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z = 0, \qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z = 0,$$

2. find the transverse fields from *hz or ez from*

$$H_x = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial x}, \quad H_y = \frac{-j\beta}{k_c^2}\frac{\partial H_z}{\partial y}, \quad E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}, \quad E_y = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}.$$

$$H_x = \frac{j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial y}, \quad H_y = \frac{-j\omega\epsilon}{k_c^2}\frac{\partial E_z}{\partial x}, \quad E_x = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial x}, \quad E_y = \frac{-j\beta}{k_c^2}\frac{\partial E_z}{\partial y}.$$

3. Apply the boundary conditions to the appropriate field components to find the unknown constants and k_c.

4. The propagation constant the wave impedance are

$$Z_{TE} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}, \qquad Z_{TM} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k},$$

Attenuation Due to Dielectric Loss

If $\boldsymbol{\alpha_d}$ **is the attenuation constant** due to dielectric loss and $\boldsymbol{\alpha c}$ **is the attenuation constant due to conductor loss**, then the total attenuation constant is $\boldsymbol{\alpha = \alpha_d + \alpha_c}$.

Attenuation caused by conductor loss can be calculated using the perturbation method, determined by the field distribution in the guide.
The attenuation due to a lossy dielectric material can be calculated from the propagation constant.

$$\gamma = \alpha_d + j\beta = \sqrt{k_c^2 - k^2}$$

$$= \sqrt{k_c^2 - \omega^2 \mu_0 \epsilon_0 \epsilon_r (1 - j \tan\delta)}.$$

$$\gamma = \sqrt{k_c^2 - k^2 + jk^2 \tan\delta}$$

$$\simeq \sqrt{k_c^2 - k^2} + \frac{jk^2 \tan\delta}{2\sqrt{k_c^2 - k^2}}$$

$$= \frac{k^2 \tan\delta}{2\beta} + j\beta,$$

k : the (real) wave number

Since $\sqrt{a^2 + x^2} \simeq a + \frac{1}{2}\left(\frac{x^2}{a}\right)$, for $x \ll a$.

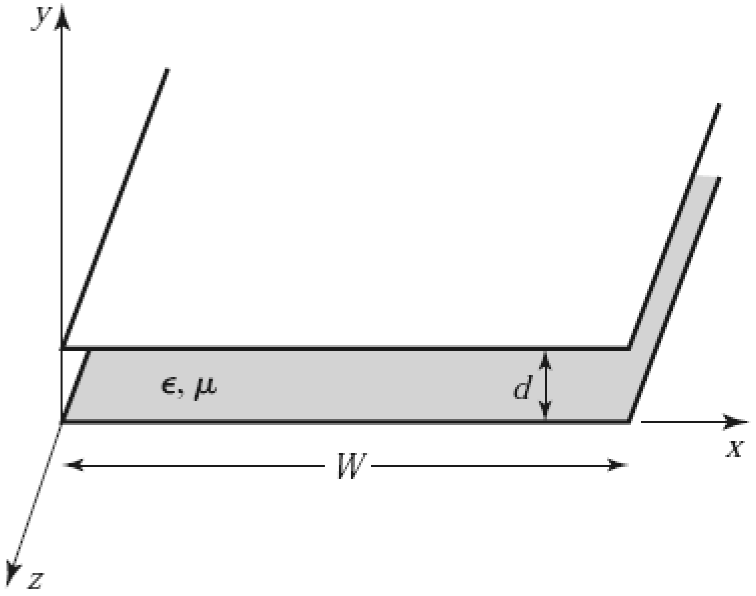
$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} \text{ Np/m (TE or TM waves).}$$

It can also be used for TEM lines, where kc = 0, by letting β = k:

$$\alpha_d = \frac{k \tan\delta}{2} \text{ Np/m (TEM waves).}$$

3.2 PARALLEL PLATE WAVEGUIDE

• The simplest type of guide that can support TM and TE modes; it can also support a TEM mode since it is formed from two flat conducting plates, or strips



the strip width, W, is assumed to be much greater than the separation, d, so that fringing fields and any x variation can be ignored.

TEM Modes

the TEM mode solution can be obtained by solving Laplace's equation

$$\nabla_t^2 \Phi(x, y) = 0, \qquad \text{for } 0 \le x \le W, \ 0 \le y \le d.$$

the boundary conditions

$$\Phi(x, 0) = 0,$$

$$\Phi(x, d) = V_o.$$

Because there is no variation in *x*,

$$\Phi(x, y) = A + By,$$

$$\Phi(x, y) = V_o y/d.$$

The transverse electric field is,

$$\bar{e}(x, y) = -\nabla_t \Phi(x, y) = -\hat{y}\frac{V_o}{d},$$

so that the total electric field is

$$\bar{E}(x, y, z) = \bar{e}(x, y)e^{-jkz} = -\hat{y}\frac{V_o}{d}e^{-jkz},$$

where $k = \omega\sqrt{\mu\epsilon}$ is the propagation constant of the TEM wave

$$\bar{H}(x, y, z) = \bar{h}(x, y)\, e^{-jkz} = \frac{1}{\eta}\hat{z} \times \bar{E}(x, y, z) = \hat{x}\frac{V_o}{\eta d}e^{-jkz},$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the medium between the parallel plates.

The voltage of the top plate with respect to the bottom plate

$$V = -\int_{y=0}^{d} E_y\, dy = V_o e^{-jkz},$$

The total current on the top plate can be found from Ampere's law or the surface current density:

$$I = \int_{x=0}^{W} \bar{J}_s \cdot \hat{z}\, dx = \int_{x=0}^{W} (-\hat{y} \times \bar{H}) \cdot \hat{z}\, dx = \int_{x=0}^{W} H_x dx = \frac{W V_o}{\eta d} e^{-jkz}.$$

the characteristic impedance is $Z_0 = \frac{V}{I} = \frac{\eta d}{W},$

which is seen to be a constant dependent only on the geometry and material parameters of the guide.

The phase velocity is also a constant:

$$v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}},$$

which is the speed of light in the material medium.

TM Modes

Since $H_z = 0$ $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z = 0,$ with $\partial/\partial x = 0$:

$$\left(\frac{\partial^2}{\partial y^2} + k_c^2\right) e_z(x, y) = 0,$$

where $k_c = \sqrt{k^2 - \beta^2}$ is the cutoff wave number, $E_z(x, y, z) = e_z(x, y)e^{-j\beta z}.$

The general solution is $e_z(x, y) = A \sin\, k_c y + B \cos k_c y,$

subject to the boundary conditions that

$$e_z(x, y) = 0, \qquad \text{at } y = 0,\ d.$$

This implies that B = 0 and $k_c d = n\pi$ for n = 0, 1, 2, 3 . . . ,

$$k_c = \frac{n\pi}{d}, \qquad n = 0,\ 1,\ 2,\ 3, \ldots.$$

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - (n\pi/d)^2}.$$

The solution for $e_z(x, y)$ *is then*

$$e_z(x, y) = A_n \sin\frac{n\pi y}{d}, \qquad E_z(x, y, z) = A_n \sin\frac{n\pi y}{d} e^{-j\beta z}.$$

The transverse field components

$$H_x = \frac{j\omega\epsilon}{k_c} A_n \cos\frac{n\pi y}{d} e^{-j\beta z},$$

$$E_y = \frac{-j\beta}{k_c} A_n \cos\frac{n\pi y}{d} e^{-j\beta z},$$

$$E_x = H_y = 0.$$

Observe that for $n = 0$, $\beta = k = \omega\sqrt{\mu\epsilon}$, and that $E_z = 0$.

The Ey and Hx fields are then constant in y, so that the TM_0 mode is actually identical to the TEM mode.

For n > 0, however, the situation is different. Each value of n corresponds to a different TM mode, denoted as the TMn mode, and each mode has its own propagation constant and field expressions.

it can be seen that β is real only when $k > kc$. Because $k = \omega\sqrt{\mu\epsilon}$

The **cutoff frequency** of the TMn mode can be found as $f_c = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{n}{2d\sqrt{\mu\epsilon}}$.

Thus, the TM mode (for *n > 0)* that propagates at the lowest frequency is the TM1 mode, with a cutoff frequency of

$$f_c = 1/2d\sqrt{\mu\epsilon}$$

At frequencies below the cutoff frequency of a given mode, the propagation constant is purely imaginary, corresponding to a rapid exponential decay of the fields. Such modes are referred to as ***cutoff modes, or evanescent modes*** (消逝模).

波导工作在截止模时，其相移常数有何特点？
波阻抗有何特点？

The wave impedance of a TM mode,

平面波波长？波导波长？截止波长？
波数？相移常数？截止波数？

$$Z_{\mathrm{TM}} = \frac{-E_y}{H_x} = \frac{\beta}{\omega\epsilon} = \frac{\beta\eta}{k},$$

Which is pure real when $f > fc$ but pure imaginary when $f < fc$.

The phase velocity is also a function of frequency:

$$v_p = \frac{\omega}{\beta},$$

and is seen to be greater than $1/\sqrt{\mu\epsilon} = \omega/k$, the speed of light in the medium, since $\beta < k$.

The guide wavelength is defined as

$$\lambda_g = \frac{2\pi}{\beta},$$ Note that $\lambda_g > \lambda = 2\pi/k$,

a cutoff wavelength for the TMn modeis $$\lambda_c = \frac{2d}{n}.$$

the time-average power passing a transverse cross section of the parallel plate guide is

$$P_o = \frac{1}{2}\mathrm{Re}\int_{x=0}^{W}\int_{y=0}^{d} \bar{E} \times \bar{H}^* \cdot \hat{z}\, dy\, dx = -\frac{1}{2}\mathrm{Re}\int_{x=0}^{W}\int_{y=0}^{d} E_y H_x^*\, dy\, dx$$

$$= \frac{W\mathrm{Re}(\beta)\omega\epsilon}{2k_c^2}|A_n|^2 \int_{y=0}^{d} \cos^2 \frac{n\pi y}{d}\, dy = \begin{cases} \dfrac{W\mathrm{Re}(\beta)\omega\epsilon\, d}{4k_c^2}|A_n|^2 & \text{for } n > 0 \\ \dfrac{W\mathrm{Re}(\beta)\omega\epsilon\, d}{2k_c^2}|A_n|^2 & \text{for } n = 0 \end{cases}$$

Po is positive and nonzero when β is real, which occurs when f > fc. When the mode is below cutoff, β is imaginary, and then Po = 0.

Conductor loss can be treated using the **perturbation method**. Thus,

$$\alpha_c = \frac{P_\ell}{2P_o},$$

where *Po* is the power flow down the guide in the absence of conductor loss, and P_l is the power dissipated per unit length in the two lossy conductors

$$P_\ell = 2\left(\frac{R_s}{2}\right)\int_{x=0}^{W} |\bar{J}_s|^2\, dx = \frac{\omega^2\epsilon^2 R_s W}{k_c^2}|A_n|^2,$$

where Rs is the surface resistivity of the conductors.

$$\alpha_c = \frac{2\omega\epsilon R_s}{\beta d} = \frac{2kR_s}{\beta\eta d} \text{ Np/m}, \qquad \text{for } n > 0.$$

the TEM mode is identical to the TM0 mode for the parallel plate waveguide, so

$$\alpha_c = \frac{R_s}{\eta d} \text{ Np/m}.$$

TE Modes

TE modes, characterized by *Ez = 0*, can also propagate in a parallel plate waveguide.

$$\partial/\partial x = 0, \quad \Rightarrow \quad \left(\frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0,$$

where $k_c = \sqrt{k^2 - \beta^2}$ is the cutoff wave number and $H_z(x, y, z) = h_z(x, y)e^{-j\beta z}$.

$$h_z(x, y) = A \sin k_c y + B \cos k_c y.$$

The boundary conditions are that *Ex = 0* at y = 0, d; Ez is identically zero for TE modes.

From $E_x = \frac{-j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$, $\quad E_x = \frac{-j\omega\mu}{k_c}(A \cos k_c y - B \sin k_c y)\ e^{-j\beta z}$,

applying the boundary conditions

$A = 0$ and $\quad k_c = \frac{n\pi}{d}, \quad n = 1, 2, 3 \ldots,$

The final solution for *Hz is then*

$$H_z(x, y) = B_n \cos \frac{n\pi y}{d} e^{-j\beta z}.$$

The transverse fields can be computed

$$E_x = \frac{j\omega\mu}{k_c} B_n \sin \frac{n\pi y}{d} e^{-j\beta z},$$

$$H_y = \frac{j\beta}{k_c} B_n \sin \frac{n\pi y}{d} e^{-j\beta z},$$

$$E_y = H_x = 0.$$

The propagation constant of the TE*n mode is given as*

$$\beta = \sqrt{k^2 - \left(\frac{n\pi}{d}\right)^2},$$

which is the same as the propagation constant of the TM*n mode.*

The cutoff frequency of the TE*n mode is* $f_c = \frac{n}{2d\sqrt{\mu\epsilon}}$,

which is also identical to that of the TM*n mode.*

The wave impedance of the TE*n mode is,* $Z_{\mathrm{TE}} = \frac{E_x}{H_y} = \frac{\omega\mu}{\beta} = \frac{k\eta}{\beta}$,

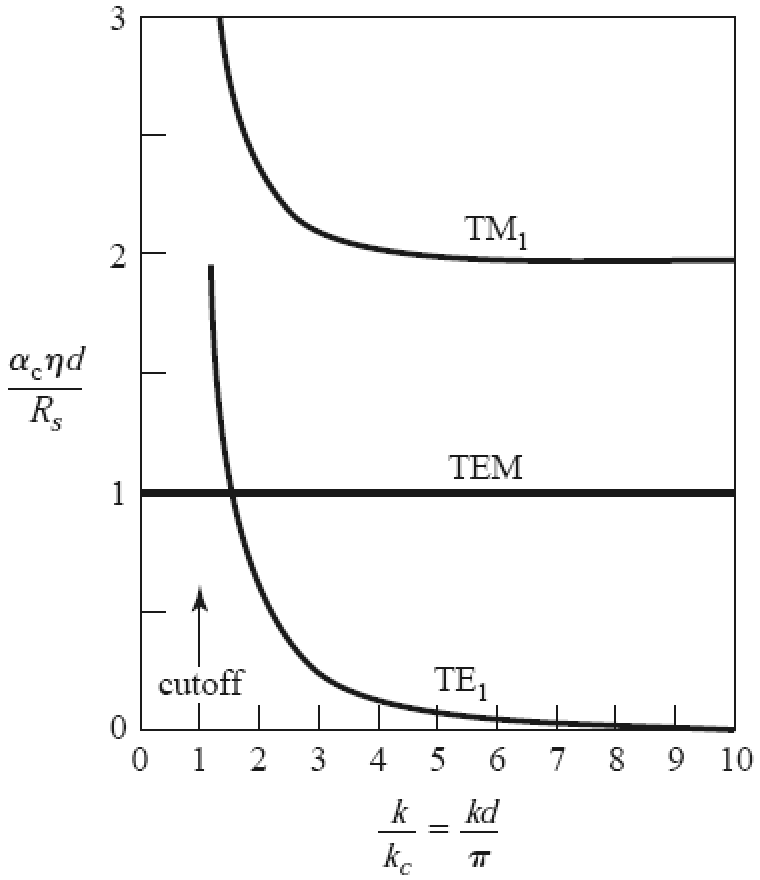
The power flow down the guide for a TE*n* mode can be calculated as

$$P_o = \frac{1}{2}\mathrm{Re}\int_{x=0}^{W}\int_{y=0}^{d} \bar{E}\times\bar{H}^*\cdot\hat{z}\,dy\,dx = \frac{1}{2}\mathrm{Re}\int_{x=0}^{W}\int_{y=0}^{d} E_x H_y^*\,dy\,dx$$
$$= \frac{\omega\mu d W}{4k_c^2}|B_n|^2\mathrm{Re}(\beta), \qquad \text{for } n > 0,$$

which is zero if the operating frequency is below the cutoff frequency (β *imaginary).*

Note that if n = 0, then Ex = Hy =0, and thus Po = 0, implying that there is no TE0 mode.

the attenuation due to conductor loss for TE modes is given by $\alpha_c = \frac{2k_c^2 R_s}{\omega\mu\beta d} = \frac{2k_c^2 R_s}{k\beta\eta d}$ Np/m.



Attenuation due to conductor loss for the TEM, TM_1, and TE_1 modes of a parallel plate waveguide.

TABLE 3.1 Summary of Results for Parallel Plate Waveguide

Quantity	TEM Mode	TM_n Mode	TE_n Mode
k	$\omega\sqrt{\mu\epsilon}$	$\omega\sqrt{\mu\epsilon}$	$\omega\sqrt{\mu\epsilon}$
k_c	0	$n\pi/d$	$n\pi/d$
β	$k=\omega\sqrt{\mu\epsilon}$	$\sqrt{k^2-k_c^2}$	$\sqrt{k^2-k_c^2}$
λ_c	∞	$2\pi/k_c=2d/n$	$2\pi/k_c=2d/n$
λ_g	$2\pi/k$	$2\pi/\beta$	$2\pi/\beta$
v_p	$\omega/k=1/\sqrt{\mu\epsilon}$	ω/β	ω/β
α_d	$(k\tan\delta)/2$	$(k^2\tan\delta)/2\beta$	$(k^2\tan\delta)/2\beta$
α_c	$R_s/\eta d$	$2kR_s/\beta\eta d$	$2k_c^2R_s/k\beta\eta d$
E_z	0	$A\sin(n\pi y/d)e^{-j\beta z}$	0
H_z	0	0	$B\cos(n\pi y/d)e^{-j\beta z}$
E_x	0	0	$(j\omega\mu/k_c)B\sin(n\pi y/d)e^{-j\beta z}$
E_y	$(-V_o/d)e^{-j\beta z}$	$(-j\beta/k_c)A\cos(n\pi y/d)e^{-j\beta z}$	0
H_x	$(V_o/\eta d)e^{-j\beta z}$	$(j\omega\epsilon/k_c)A\cos(n\pi y/d)e^{-j\beta z}$	0
H_y	0	0	$(j\beta/k_c)B_n\sin(n\pi y/d)e^{-j\beta z}$
Z	$Z_{\text{TEM}}=\eta d/W$	$Z_{\text{TM}}=\beta\eta/k$	$Z_{\text{TE}}=k\eta/\beta$

3.3 RECTANGULAR WAVEGUIDE

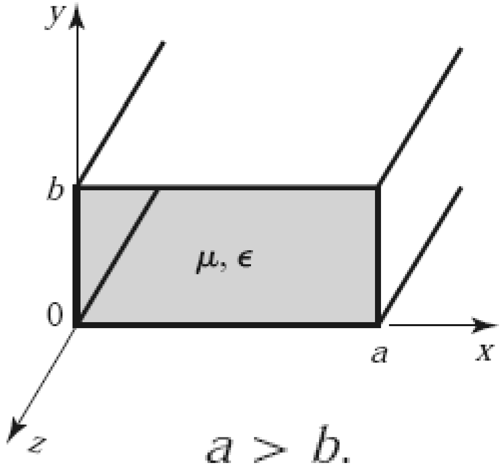
Rectangular waveguides were one of the earliest types of transmission lines used to transport microwave signals, and they are still used for many applications， such as couplers, detectors, isolators, attenuators, and slotted lines with the waveguide bands from 1 to 220 GHz.

Because of the trend toward miniaturization and integration, most modern microwave circuitry is fabricated using planar transmission lines such as microstrips and stripline rather than waveguides.

There is, however, still a need for waveguides in many cases, including high-power systems, millimeter wave applications, satellite systems, and some precision test applications.

TE Modes

The hollow rectangular waveguide can propagate TM and TE modes but not TEM waves since only one conductor is present.



$a > b.$

TE waveguide modes are characterized by fields with *Ez = 0, while Hz must satisfy* the reduced wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) h_z(x, y) = 0,$$

with $H_z(x, y, z) = h_z(x, y) e^{-j\beta z}$;

$k_c = \sqrt{k^2 - \beta^2}$ is the cutoff wave number.

The partial differential equation can be solved by the method of separation of variables by letting

$$h_z(x, y) = X(x) Y(y)$$

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + k_c^2 = 0.$$

each of the terms must be equal to a constant,

$$\frac{d^2X}{dx^2} + k_x^2 X = 0,$$

$$\frac{d^2Y}{dy^2} + k_y^2 Y = 0,$$

$$k_x^2 + k_y^2 = k_c^2.$$

The general solution for *hz can then be written as*

$$h_z(x, y) = (A\cos k_x x + B\sin k_x x)(C\cos k_y y + D\sin k_y y).$$

$$e_x = \frac{-j\omega\mu}{k_c^2} k_y (A\cos k_x x + B\sin k_x x)(-C\sin k_y y + D\cos k_y y),$$

$$e_y = \frac{j\omega\mu}{k_c^2} k_x (-A\sin k_x x + B\cos k_x x)(C\cos k_y y + D\sin k_y y).$$

apply the boundary conditions

$$e_x(x, y) = 0, \qquad \text{at } y = 0, b,$$

$$e_y(x, y) = 0, \qquad \text{at } x = 0, a.$$

$$D = 0, \text{ and } k_y = n\pi/b \text{ for } n = 0, 1, 2. \ldots$$

$$B = 0 \text{ and } k_x = m\pi/a \text{ for } m = 0, 1, 2. \ldots$$

The final solution for *Hz is then*

$$H_z(x, y, z) = A_{mn} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z},$$

The transverse field components of the TE*mn mode*

$$E_x = \frac{j\omega\mu n\pi}{k_c^2 b} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z},$$

$$E_y = \frac{-j\omega\mu m\pi}{k_c^2 a} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z},$$

$$H_x = \frac{j\beta m\pi}{k_c^2 a} A_{mn} \sin\frac{m\pi x}{a} \cos\frac{n\pi y}{b} e^{-j\beta z},$$

$$H_y = \frac{j\beta n\pi}{k_c^2 b} A_{mn} \cos\frac{m\pi x}{a} \sin\frac{n\pi y}{b} e^{-j\beta z}.$$

The propagation constant is $$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2},$$

which is seen to be real, corresponding to a **propagating mode**,

$$k > k_c = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}.$$

Each mode (each combination of *m and n)* has a **cutoff frequency** f_{cmn}

$$f_{c_{mn}} = \frac{k_c}{2\pi\sqrt{\mu\epsilon}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}.$$

The mode with the lowest cutoff frequency is called the dominant mode; $a > b$, the lowest cutoff frequency occurs for the TE10 (m = 1, n = 0) mode:

$$f_{c_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}}.$$

Thus the TE10 mode is the dominant TE mode and, as we will see, the overall dominant mode of the rectangular waveguide.

Since E and H are all zero if both m = n = 0; there is no TE_{00} mode.

If more than one mode is propagating, the waveguide is said to be ***overmoded***.

$$Z_{\mathrm{TE}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{k\eta}{\beta},$$

where $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the material filling the waveguide.

Note that Z_{TE} *is real when β is real (a propagating mode) but is imaginary when β is imaginary* (a cutoff mode).

The guide wavelength is defined as the distance between two equal-phase planes

$$\lambda_g = \frac{2\pi}{\beta} > \frac{2\pi}{k} = \lambda,$$

λ, the wavelength of a plane wave in the medium filling the guide.

The phase velocity is greater than the speed of light (plane wave) in the medium.

$$v_p = \frac{\omega}{\beta} > \frac{\omega}{k} = 1/\sqrt{\mu\epsilon},$$

In the vast majority of waveguide applications **the operating frequency and guide dimensions are chosen so that only the dominant TE10 mode will propagate**.

for the TE_{10} mode fields:

$$H_z = A_{10} \cos \frac{\pi x}{a} e^{-j\beta z},$$

$$E_y = \frac{-j\omega\mu a}{\pi} A_{10} \sin \frac{\pi x}{a} e^{-j\beta z},$$

$$H_x = \frac{j\beta a}{\pi} A_{10} \sin \frac{\pi x}{a} e^{-j\beta z},$$

$$E_x = E_z = H_y = 0.$$

The cutoff wave number and propagation constant for the TE10 mode are,

$$k_c = \pi / a,$$

$$\beta = \sqrt{k^2 - (\pi/a)^2}.$$

The power flow down the guide for the TE10 mode can be calculated as

$$
\begin{aligned}
P_{10} &= \frac{1}{2}\mathrm{Re}\int_{x=0}^{a}\int_{y=0}^{b} \bar{E}\times\bar{H}^{*}\cdot\hat{z}\,dy\,dx \\
&= \frac{1}{2}\mathrm{Re}\int_{x=0}^{a}\int_{y=0}^{b} E_y H_x^{*}\,dy\,dx \\
&= \frac{\omega\mu a^2}{2\pi^2}\mathrm{Re}(\beta)|A_{10}|^2\int_{x=0}^{a}\int_{y=0}^{b}\sin^2\frac{\pi x}{a}\,dy\,dx \\
&= \frac{\omega\mu a^3|A_{10}|^2 b}{4\pi^2}\mathrm{Re}(\beta).
\end{aligned}
$$

Note that this result gives nonzero real power only when β *is real, corresponding to* a propagating mode.

The power lost per unit length due to finite wall conductivity is,

$$P_\ell = \frac{R_s}{2}\int_C |\bar{J}_s|^2 \, d\ell,$$

where Rs is the wall surface resistance, and the integration contour C encloses the inside perimeter of the guide walls.

There are surface currents on all four walls, but from symmetry the currents on the top and bottom walls are identical, as are the currents on the left and right side walls. So we can compute the power lost in the walls at *x = 0 and y = 0* and double their sum to obtain the total power loss. The surface current on the *x = 0 (left)* wall is

The surface current on the *x = 0 (left)* wall is

$$\bar{J}_s = \hat{n} \times \bar{H}|_{x=0} = \hat{x} \times \hat{z} H_z|_{x=0} = -\hat{y} H_z|_{x=0} = -\hat{y} A_{10} e^{-j\beta z},$$

the surface current on the y = 0 (bottom) wall is

$$\bar{J}_s = \hat{n} \times \bar{H}|_{y=0} = \hat{y} \times (\hat{x} H_x|_{y=0} + \hat{z} H_z|_{y=0})$$

$$= -\hat{z}\frac{j\beta a}{\pi} A_{10} \sin \frac{\pi x}{a} e^{-j\beta z} + \hat{x} A_{10} \cos \frac{\pi x}{a} e^{-j\beta z}.$$

$$P_\ell = R_s \int_{y=0}^{b} |J_{sy}|^2 dy + R_s \int_{x=0}^{a} \left[|J_{sx}|^2 + |J_{sz}|^2 \right] dx$$

$$= R_s |A_{10}|^2 \left(b + \frac{a}{2} + \frac{\beta^2 a^3}{2\pi^2} \right).$$

The attenuation due to conductor loss for the TE10 mode is then

$$\alpha_c = \frac{P_\ell}{2 P_{10}} = \frac{2\pi^2 R_s (b + a/2 + \beta^2 a^3 / 2\pi^2)}{\omega \mu a^3 b \beta}$$

$$= \frac{R_s}{a^3 b \beta k \eta} (2b\pi^2 + a^3 k^2) \text{ Np/m.}$$

TM Modes

TM modes are characterized by fields with *Hz = 0, while Ez* must satisfy the reduced wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_c^2\right) e_z(x, y) = 0,$$

with $E_z(x, y, z) = e_z(x, y)e^{-j\beta z}$ and $k_c^2 = k^2 - \beta^2$.

The general solution is $e_z(x, y) = (A \cos k_x x + B \sin k_x x)(C \cos k_y y + D \sin k_y y).$

The boundary conditions can be applied directly to e_z

$$e_z(x, y) = 0, \qquad \text{at } x = 0, a,$$

$$e_z(x, y) = 0, \qquad \text{at } y = 0, b.$$

$A = 0$ and $k_x = m\pi/a$ for $m = 1, 2, 3\ldots$

$C = 0$ and $k_y = n\pi/b$ for $n = 1, 2, 3\ldots$

$$E_z(x, y, z) = B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z},$$

The transverse field components for the TM*mn mode*

$$E_x = \frac{-j\beta m\pi}{ak_c^2} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z},$$

$$E_y = \frac{-j\beta n\pi}{bk_c^2} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z},$$

$$H_x = \frac{j\omega\epsilon n\pi}{bk_c^2} B_{mn} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z},$$

$$H_y = \frac{-j\omega\epsilon m\pi}{ak_c^2} B_{mn} \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}.$$

the propagation constant is

$$\beta = \sqrt{k^2 - k_c^2} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

The cutoff frequencies for the TM*mn* modes are also the same as those of the TEmn modes.

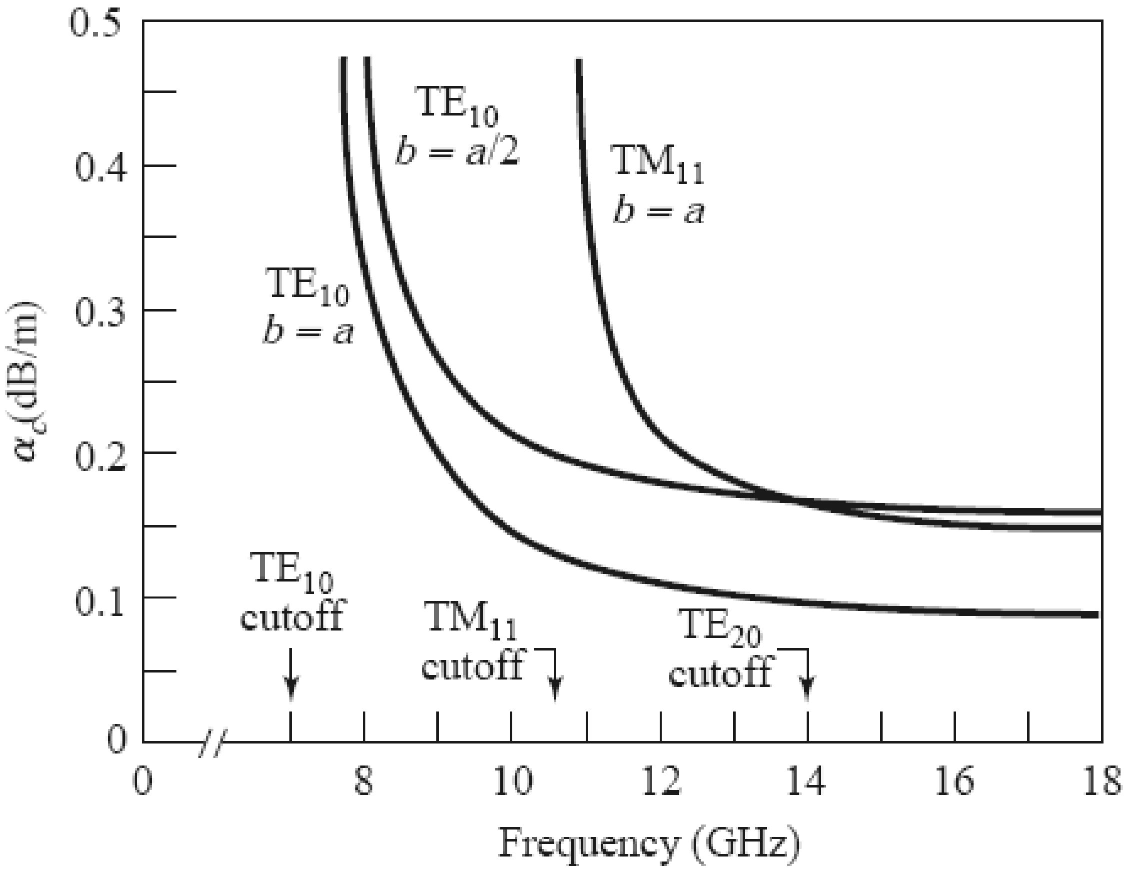
there is no TM00, TM01, or TM10 mode, and **the lowest order TM mode to propagate (lowest fc) is the TM11 mode, having a cutoff frequency of**

$$f_{c_{11}} = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2},$$

which is seen to be larger than fc10 , the cutoff frequency of the TE10 mode.

The wave impedance

$$Z_{\text{TM}} = \frac{E_x}{H_y} = \frac{-E_y}{H_x} = \frac{\beta\eta}{k}.$$

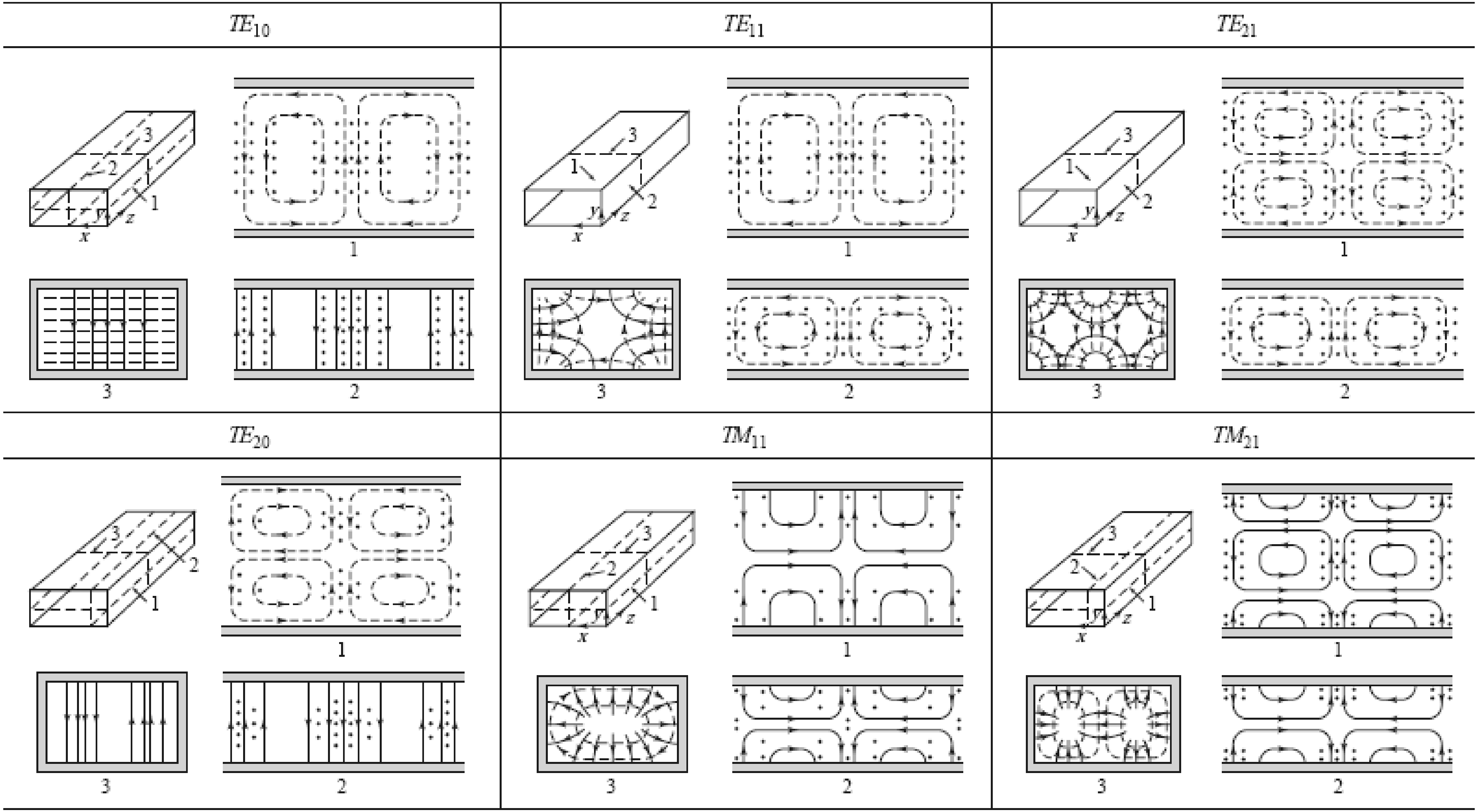


Attenuation of various modes in a rectangular brass waveguide with *a = 2.0 cm.*

Summary of Results for Rectangular Waveguide

Quantity	TE_{mn} Mode	TM_{mn} Mode
k	$\omega\sqrt{\mu\epsilon}$	$\omega\sqrt{\mu\epsilon}$
k_c	$\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$	$\sqrt{(m\pi/a)^2 + (n\pi/b)^2}$
β	$\sqrt{k^2 - k_c^2}$	$\sqrt{k^2 - k_c^2}$
λ_c	$\frac{2\pi}{k_c}$	$\frac{2\pi}{k_c}$
λ_g	$\frac{2\pi}{\beta}$	$\frac{2\pi}{\beta}$
v_p	$\frac{\omega}{\beta}$	$\frac{\omega}{\beta}$
α_d	$\frac{k^2 \tan\delta}{2\beta}$	$\frac{k^2 \tan\delta}{2\beta}$

Quantity	TE_{mn} Mode	TM_{mn} Mode
E_z	0	$B \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$
H_z	$A \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$	0
E_x	$\frac{j\omega\mu n\pi}{k_c^2 b} A \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$	$\frac{-j\beta m\pi}{k_c^2 a} B \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$
E_y	$\frac{-j\omega\mu m\pi}{k_c^2 a} A \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$	$\frac{-j\beta n\pi}{k_c^2 b} B \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$
H_x	$\frac{j\beta m\pi}{k_c^2 a} A \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$	$\frac{j\omega\epsilon n\pi}{k_c^2 b} B \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} e^{-j\beta z}$
H_y	$\frac{j\beta n\pi}{k_c^2 b} A \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$	$\frac{-j\omega\epsilon m\pi}{k_c^2 a} B \cos \frac{m\pi x}{a} \sin \frac{n\pi y}{b} e^{-j\beta z}$
Z	$Z_{\text{TE}} = \frac{k\eta}{\beta}$	$Z_{\text{TM}} = \frac{\beta\eta}{k}$



- - -H, ——E

EXAMPLE 3.1 CHARACTERISTICS OF A RECTANGULAR WAVEGUIDE

Consider a length of Teflon-filled, copper K-band rectangular waveguide having dimensions *a = 1.07 cm and b = 0.43 cm.* Find the cutoff frequencies of the first five propagating modes. If the operating frequency is 15 GHz, find the attenuation due to dielectric and conductor losses.

Solution

From Appendix G, for Teflon, *r = 2.08 and tan δ = 0.0004*. The cutoff frequencies are given by

$$f_{c_{mn}} = \frac{c}{2\pi\sqrt{\epsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}.$$

Mode	m	n	f_c(GHz)
TE	1	0	9.72
TE	2	0	19.44
TE	0	1	24.19
TE, TM	1	1	26.07
TE, TM	2	1	31.03

APPENDIX G

DIELECTRIC CONSTANTS AND LOSS TANGENTS FOR SOME MATERIALS

Material	Frequency	ϵ_r	$\tan\delta$ (25°C)
Silicon	10 GHz	11.9	0.004
Styrofoam (103.7)	3 GHz	1.03	0.0001
Teflon	10 GHz	2.08	0.0004

Thus the TE10, TE20, TE01, TE11, and TM11 modes will be the first five modes to propagate.

At 15 GHz, the propagation constant for the TE10 mode

$$\beta = \sqrt{\left(\frac{2\pi f\sqrt{\epsilon_r}}{c}\right)^2 - \left(\frac{\pi}{a}\right)^2} = \sqrt{k^2 - \left(\frac{\pi}{a}\right)^2} = 345.1\ \mathrm{m}^{-1}.$$

Degenerate mode (简并模式): different modes have the same cutoff number, such as TEmn and TMmn the attenuation due to dielectric loss is

$$\alpha_d = \frac{k^2 \tan\delta}{2\beta} = 0.119 \text{ Np/m} = 1.03 \text{ dB/m.}$$

The surface resistivity of the copper walls is ($\sigma = 5.8 \times 10^7$ S/m)

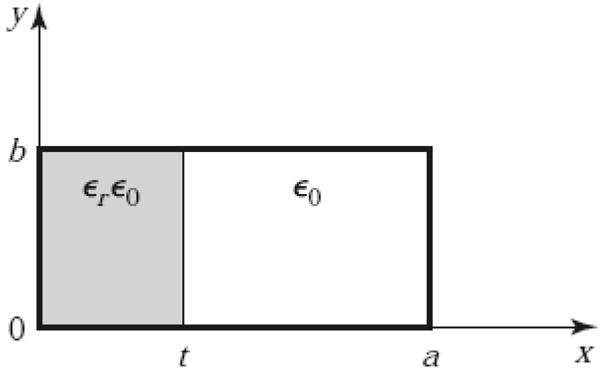
$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 0.032\ \Omega,$$

the attenuation due to conductor loss,

$$\alpha_c = \frac{R_s}{a^3 b\beta k\eta}(2b\pi^2 + a^3 k^2) = 0.050 \text{ Np/m} = 0.434 \text{ dB/m.}$$

*TE_{m0} Modes of a Partially Loaded Waveguide

in some cases of practical interest (such as impedance matching or phase-shifting sections) a waveguide is used with a partial dielectric filling.



Since the geometry is uniform in the y direction and n = 0, the TEm0 modes have no y dependence.

$$\left(\frac{\partial^2}{\partial x^2} + k_d^2\right) h_z = 0, \qquad \text{for } 0 \le x \le t,$$

$$\left(\frac{\partial^2}{\partial x^2} + k_a^2\right) h_z = 0, \qquad \text{for } t \le x \le a,$$

where k_d and k_a are the cutoff wave numbers for the dielectric and air regions,

$$\beta = \sqrt{\epsilon_r k_0^2 - k_d^2},$$

$$\beta = \sqrt{k_0^2 - k_a^2}.$$

the propagation constant, *β, must be the same in* both regions to ensure phase matching (see Section 1.8) of the fields along the interface at *x* = *t.*

The solutions

$$h_z = \begin{cases} A\cos k_d x + B\sin k_d x & \text{for } 0 \le x \le t \\ C\cos k_a(a-x) + D\sin k_a(a-x) & \text{for } t \le x \le a, \end{cases}$$

We need $\hat{y}$ and $\hat{z}$ electric and magnetic field components to apply the boundary conditions at $x = 0$, t, and a. $E_z = 0$ for TE modes, and $H_y = 0$ since $\partial/\partial y = 0$. E_y is found

$$e_y = \begin{cases} \dfrac{j\omega\mu_0}{k_d}(-A\sin k_d x + B\cos k_d x) & \text{for } 0 \le x \le t \\ \dfrac{j\omega\mu_0}{k_a}[C\sin k_a(a-x) - D\cos k_a(a-x)] & \text{for } t \le x \le a. \end{cases}$$

To satisfy the boundary conditions that Ey = 0 at x = 0 and x = a requires that B = D = 0.

We next enforce continuity of tangential fields (*Ey , Hz) at x = t.*

$$\frac{-A}{k_d}\sin k_d t = \frac{C}{k_a}\sin k_a(a-t),$$

$$A\cos k_d t = C\cos k_a(a-t).$$

this is a homogeneous set of equations （齐次方程）, the determinant （行列式） must vanish

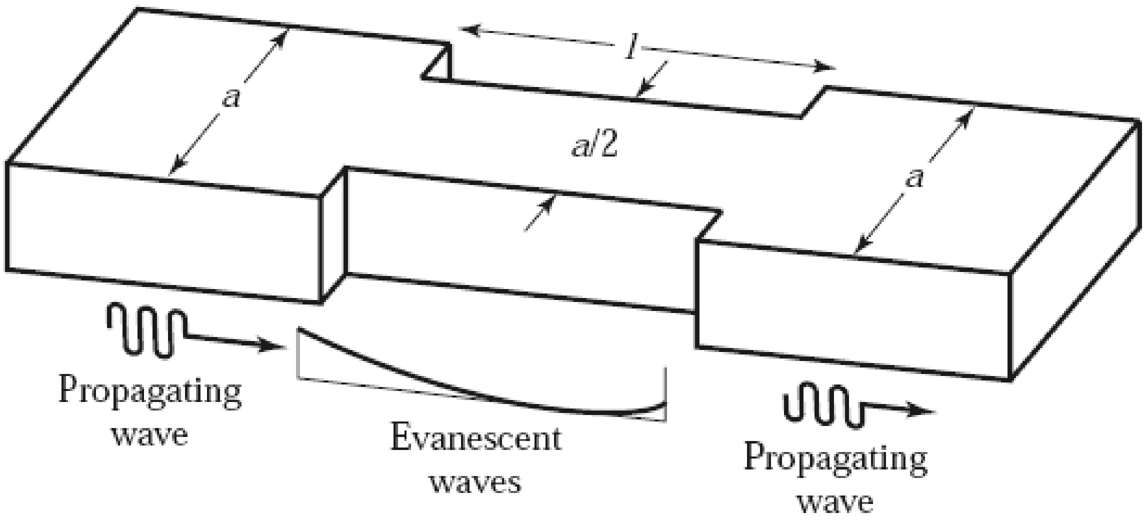
$$k_a \tan k_d t + k_d \tan k_a(a-t) = 0.$$

There is an infinite number of solutions, corresponding to the propagation constants of the TEm0 modes.

Homework

3.3 Calculate the attenuation due to conductor loss for the TE_n mode of a parallel plate waveguide.

3.6 An attenuator can be made using a section of waveguide operating below cutoff, as shown in the accompanying figure. If $a = 2.286$ cm and the operating frequency is 12 GHz, determine the required length of the below-cutoff section of waveguide to achieve an attenuation of 100 dB between the input and output guides. Ignore the effect of reflections at the step discontinuities.



3.8 Derive the expression for the attenuation of the TM_{mn} mode of a rectangular waveguide due to imperfectly conducting walls.